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REALIZED MEASURES OF VOLATILITY

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Exponential GARCH Modeling with Realized Measures of Volatility

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Abstract

We introduce the Realized Exponential GARCH model that can utilize multiple realized volatility measures for the modeling of a return series. The model specifies the dynamic properties of both returns and realized measures, and is characterized by a flexible modeling of the dependence between returns and volatility. We apply the model to DJIA stocks and an exchange traded fund that tracks the S&P 500 index and find that specifications with multiple realized measures dominate those that rely on a single realized measure. The empirical analysis suggests some convenient simplifications and highlights the advantages of the new specification.

Keywords: EGARCH; High Frequency Data; Realized Variance; Leverage Effect.

JEL Classification: C10; C22; C80

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1 Introduction

The Realized GARCH framework by Hansen, Huang, and Shek (2012) provides a structure for the joint modeling of returns $\{r_t\}$ and realized measures of volatility $\{x_t\}$. In this paper, we introduce a new variant within this framework, called the Realized Exponential GARCH model. Key features of this model include: (1) The ability to incorporate multiple realized measures of volatility, such as the realized variance and the daily range; (2) A flexible modeling of the dependence between returns and volatility, which is known to be empirically important - a result that is confirmed in our empirical analysis of the DJIA stocks and the exchange traded index fund, SPY. The present paper also contributed to the literature with a number of empirical results. We undertake an extensive empirical analysis that motivates several refinements and simplifications of the model. We compare a range of realized measures and show that the log-likelihood for daily returns can be improved by the use of multiple realized measures. This is true in-sample and out-of-sample. The empirical analysis also highlights the advantages of the new specification, and we provide theoretical insight about the underlying reasons for this.

GARCH models are time-series models that specify the conditional distribution of the next period's observation - typically a return on some financial asset. The key variable is the conditional variance which is defined by past variables. Conventional GARCH models, such as the ARCH by Engle (1982) and GARCH by Bollerslev (1986), rely exclusively on daily returns (typically squared returns) for the modeling of volatility. A shortcoming of conventional GARCH models is the fact that returns are rather weak signals about the level of volatility. This makes GARCH models poorly suited for situations where volatility "jumps" to a new level over a short period of time. In such a situation, a GARCH model will be slow at 'catching up', so that it takes several periods for the conditional variance (implied by the GARCH model) to reach the new level, see Andersen et al. (2003) for discussion on this. Incorporating realized measures into GARCH models can greatly alleviate this problem.

A wide range of realized measures of volatility have been proposed in the literature since Andersen and Bollerslev (1998) showed that such measures can be very useful for the evaluation of volatility models. Realized measures of volatility, such as the popular realized variance, are computed from high-frequency data, see Andersen, Bollerslev, Diebold, and Labys (2001) and Barndorff-Nielsen and Shephard (2002). The realized variance is sensitive to market microstructure noise, which has motivated the development of robust realized measures, such as the two-scale and multi-scale estimator by Zhang et al. (2005) and Zhang (2006), respectively, the realized kernel by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), the realized range by Christensen and Podolskij (2007), see also Andersen et al. (2008), Hansen and Horel (2009) and references therein. Because realized measures are far more informative about the current level of volatility than the squared return, it can be very useful to include such in the modeling of volatility. The economic and statistical gains from incorporating realized measures in volatility models are typically found to be large, see e.g. Christoffersen et al. (2010) and Dobrev and Szerszen (2010).

Following the early work by Andersen and Bollerslev (1998) that had documented the value of using realized measures in the evaluation of volatility models, Engle (2002) explored the idea of including the realized variance as a predetermined variable in the GARCH equation, and found it to be highly significant and greatly enhancing the empirical fit, see also Forsberg and Bollerslev (2002). The first *complete* model (complete in the sense of specifying the dynamic properties of all observed time-series) was introduced by Engle and Gallo (2006), who referred to the model as a Multiplicative Error Model (MEM). The MEM framework operates with multiple latent volatility processes - one for returns and one for each of the realized measures. This structure is also the basis for variant proposed by Shephard and Sheppard (2010), who refer to their model as the HEAVY model. See also, Visser (2011) and Chen et al. (2011).

The Realized GARCH framework takes a different approach. Instead of introducing additional latent variables to the model, the Realized GARCH framework is based on measurement equations that tie the realized measure to the latent conditional variance. This has the advantage for the explicit modeling of leverage effect, and circumvents the need for additional latent volatility processes. The idea of using a measurement equation to tie the realized measure to the latent volatility goes back to Takahashi et al. (2009), who used it in the context of stochastic volatility models. Additional MEM specifications have been explored and developed in Cipollini et al. (2009) and Brownless and Gallo (2010).

To illustrate the structure of a Realized EGARCH model and how it compares with conventional models, we give a brief preview of our empirical result. Below we have estimated the GARCH(1,1) by Bollerslev (1986), the EGARCH(1,1) by Nelson (1991), and a Realized EGARCH model with daily returns, $\{r_t\}$, on the S&P500 index over the sample period spanning January 1, 2002 to December 31, 2005. The full details will be presented in Section 4. The three models have the same *return equation*, $r_t = \mu + \sqrt{h_t} z_t$ with $z_t \sim \text{iid}N(0, 1)$, but their specifications for dynamic properties of the conditional variance, $h_t = \text{var}(r_t | \mathcal{F}_{t-1})$, differ in important ways. For the GARCH(1,1) and EGARCH(1,1) we estimated their GARCH equations to,

$$h_{t+1} = \begin{matrix} 0.004 \\ (0.003) \end{matrix} + \begin{matrix} 0.995 h_t \\ (0.014) \end{matrix} + \begin{matrix} 0.051 (r_t^2 - h_t) \\ (0.013) \end{matrix},$$

and

$$\log h_{t+1} = \begin{matrix} -0.026 \\ (0.020) \end{matrix} + \begin{matrix} 0.994 \log h_t \\ (0.002) \end{matrix} - \begin{matrix} 0.071 z_t \\ (0.013) \end{matrix} + \begin{matrix} 0.031 (z_t^2 - 1) \\ (0.02) \end{matrix},$$

respectively, where the numbers in brackets are standard error. In comparison, a Realized EGARCH model that utilizes two realized measures, $x_{RK,t}$ and $x_{DR,t}$, leads to the following estimated GARCH equation,

$$\log h_{t+1} = \begin{matrix} -0.006 \\ (0.005) \end{matrix} + \begin{matrix} 0.977 \log h_t \\ (0.005) \end{matrix} - \begin{matrix} 0.111 z_t \\ (0.010) \end{matrix} + \begin{matrix} 0.042 (z_t^2 - 1) \\ (0.001) \end{matrix} + \begin{matrix} 0.165 u_{RK,t} \\ (0.031) \end{matrix} + \begin{matrix} 0.084 u_{DR,t} \\ (0.017) \end{matrix},$$

where $x_{\text{RK},t}$ and $x_{\text{DR},t}$ denote a realized kernel for day t and the daily (squared) range, respectively, and where u_{RK} and u_{DR} are given from the two measurement equations:

$$\begin{aligned}\log x_{\text{RK},t} &= \underset{(0.046)}{-0.360} + \log h_t - \underset{(0.011)}{0.010}z_t + \underset{(0.007)}{0.027}(z_t^2 - 1) + u_{\text{RK},t}, \\ \log x_{\text{DR},t} &= \underset{(0.041)}{-0.440} + \log h_t - \underset{(0.019)}{0.066}z_t + \underset{(0.017)}{0.239}(z_t^2 - 1) + u_{\text{DR},t},\end{aligned}$$

with $(u_{\text{RK}}, u_{\text{DR}})'$ specified to be iid and Gaussian distributed with mean zero and a variance-covariance matrix, which is estimated to be

$$\hat{\Sigma} = \begin{pmatrix} 0.133 & 0.627 \\ 0.627 & 0.429 \end{pmatrix}.$$

The realized measures contribute to modeling the volatility dynamics through the coefficients for u_{RK} and u_{DR} in the GARCH equations. These coefficients are both significant. The structure of the estimated covariance matrix, $\hat{\Sigma}$, shows (not surprisingly) that the realized kernel is a far more accurate measurement of the conditional variance than is the daily range.

	In-Sample Log-Likelihood	Out-of-Sample Log-Likelihood
GARCH	-1330.05	-871.86
Exponential GARCH	-1308.88	-876.08
Realized Exponential GARCH	-1305.77	-855.88

Table 1: In-sample and out-of-sample log-likelihoods.

The real benefits of including realized measures in the GARCH modeling are revealed by comparing the value of the log-likelihood function for daily returns. In Table 1 we present the value of the log-likelihood functions for the three specifications, both in-sample (i.e. the sample period with the parameter estimates were computed from) and for the out-of-sample period that spans January 1, 2006 to August 29, 2009. The latter is the log-likelihood computed for the out-of-sample data using the in-sample parameter estimates. The Realized EGARCH model dominates the GARCH and EGARCH models in terms of the in-sample log-likelihood, albeit the latter only falls three log-likelihood units behind. Out-of-sample we observe a bigger difference, and the Realized EGARCH is 25-30 units better than the two conventional GARCH specifications. This highlights the benefits of incorporating realized measures in models for return volatility.

This paper is organized as follows. We introduce the Realized Exponential GARCH model in Section 2, and quasi maximum likelihood estimation (QMLE) and inference are discussed in Section 3. Our empirical results are mainly presented in Section 4, and Section 5 compares the proposed specification to that in Hansen, Huang, and Shek (2012) in terms of both theoretical and empirical properties. We make some concluding remarks in Section 6.

2 Realized EGARCH Model

In this section, we introduce the Realized Exponential GARCH model (in short, Realized EGARCH). We use this terminology because the model shares some features with the EGARCH model by Nelson (1991). The Realized EGARCH model is well suited for the case with multiple realized measures of volatility, in which case we let x_t denote a vector of realized measures, $x_t = (x_{1,t}, \dots, x_{K,t})'$. Moreover, the model permits a more flexible modeling of the joint dependence of returns and volatility. The latter is shown to be very useful in our empirical analysis. The vector of realized measures, x_t , may include the realized variance, bipower variation, daily range, squared return, and robust measures such as the realized kernel.

Let $\{\mathcal{F}_t\}$ be a filtration so that (r_t, x_t) is adapted to \mathcal{F}_t , and define the conditional mean, $\mu_t = E(r_t|\mathcal{F}_{t-1})$, and conditional variance, $h_t = \text{var}(r_t|\mathcal{F}_{t-1})$. The Realized EGARCH model with K realized measures is given by the following equations

$$\begin{aligned} r_t &= \mu_t + \sqrt{h_t} z_t, \\ \log h_t &= \omega + \beta \log h_{t-1} + \tau(z_{t-1}) + \gamma' u_{t-1}, \\ \log x_{k,t} &= \xi_k + \varphi_k \log h_t + \delta_{(k)}(z_t) + u_{k,t}, \quad k = 1, \dots, K. \end{aligned}$$

We refer to these as the *return equation*, the *GARCH equation*, and the *measurement equation(s)*, respectively. We discuss these three equations in greater details below. In our quasi likelihood analysis we adopt a Gaussian specification, $z_t \sim N(0, 1)$ and $u_t \sim N(0, \Sigma)$, where z_t and $u_t = (u_{1,t}, \dots, u_{K,t})'$ are mutually and serially independent. The leverage functions, $\tau(z)$ and $\delta_{(k)}(z)$, $k = 1, \dots, K$, play an important role in order to make the independence between z_t and u_t realistic in practice. In the empirical analysis, we adopt quadratic form for the leverage functions,

$$\tau(z) = \tau_1 z + \tau_2 (z^2 - 1) \quad \text{and} \quad \delta_{(k)}(z) = \delta_{k,1} z + \delta_{k,2} (z^2 - 1), \quad k = 1, \dots, K.$$

The leverage functions facilitate a modeling of the dependence between return shocks and volatility shocks, which is empirically important, and the volatility shock, $v_t = E(\log h_{t+1}|\mathcal{F}_t) - E(\log h_{t+1}|\mathcal{F}_{t-1})$, is given by $v_t = \tau(z_t) + \gamma' u_t$ in this model.

The return equation is standard in GARCH models. The conditional mean, μ_t , may be modeled with a GARCH-in-mean specification or simply a constant. In fact, imposing the constraint $\mu_t = 0$ can result in better out-of-sample fit relative to a model based on an unrestricted μ . That is indeed what we find in our empirical analysis.

The GARCH equation plays a central role in models of the conditional variance, and a key feature of the Realized EGARCH model is the presence of a leverage function, $\tau(z_{t-1})$, in the GARCH equation.¹ The EGARCH model is often based on $\tau(z) = az + b|z|$. We prefer a polynomial specification for τ for

¹The Realized GARCH model in Hansen et al. (2012) only includes a leverage function in the measurement equation.

empirical reasons and because the likelihood analysis is simplified by the fact that $\tau(z)$ is differentiable at zero. Another key feature of the Realized EGARCH model is the last term in the GARCH equation. This term, $\gamma' u_{t-1}$, is the main channel by which the realized measures drive expectations of future volatility up or down. The fact that u_t is K -dimensional enables us to utilize multiple realized measures of volatility. It is worth noting that this GARCH equation specifies an ARMA(1,1) model for the conditional variance, with innovations given by $\tau(z_{t-1}) + \gamma' u_{t-1}$. Hence, the parameter β summarizes the persistence of volatility, whereas γ represents how informative the realized measures are about future volatility.

The measurement equation defines the link between the (ex-post) realized measures of volatility and the (ex-ante) conditional variance. An ex-post measure of volatility will differ from the conditional variance for a number of reasons. One source for this discrepancy is due to the fact that realized measures are not perfect measures of volatility. Empirical measures entail sampling error and even the most accurate realized measures are known to have non-negligible sampling error in practice. Another source for the discrepancy is due to difference between ex-post volatility and ex-ante volatility, which we can label the volatility shock.

The three equations fully characterize the dynamic properties of returns and realized measures of volatility. So that the model is complete in the sense that it fully specifies the dynamic properties of both returns and the realized measures.

3 Estimation and Inference

In this section, we discuss estimation and inference within the quasi-maximum likelihood framework. The analysis largely follows that in Hansen et al. (2012), but the fact that the present framework allows for multiple realized measures and the introduction of a leverage function in the GARCH equation require some modifications and extensions of the analysis in Hansen et al. (2012).

We adopt a Gaussian specification by assuming $z_t \sim \text{iid}N(0, 1)$ and $u_t \sim \text{iid}N(0, \Sigma)$, with z_t and u_t independent. Express the leverage functions as $\tau(z_t) = \tau' a(z_t)$ and $\delta_{(k)}(z_t) = \delta'_k b(z_t)$, $k = 1, \dots, K$, where $a(z_t)$ and $b(z_t)$ are known functions of z_t . In our empirical analysis we use $a_t = b_t = (z_t, z_t^2 - 1)'$. The initial value of the conditional variance, h_1 , is treated as an unknown parameter, which is a common approach when estimating GARCH models. So the parameters are

$$\theta = (\tilde{h}_1, \mu, \lambda', \psi'_1, \dots, \psi'_K)' \quad \text{and} \quad \Sigma,$$

where

$$\lambda = (\omega, \beta, \tau', \gamma')' \quad \text{and} \quad \psi_k = (\xi_k, \varphi_k, \delta'_k)', \quad k = 1, \dots, K.$$

To simplify the notation we write $\tilde{h}_t = \log h_t$ and $\tilde{x}_{k,t} = \log x_{k,t}$, and define

$$g_t = (1, \tilde{h}_t, a'_t, u'_t)', \quad \text{and} \quad m_t = (1, \tilde{h}_t, b'_t)',$$

where $a_t = a(z_t)$ and $b_t = b(z_t)$. This enables us to express the GARCH and measurement equations as

$$\tilde{h}_t = \lambda' g_{t-1} \quad \text{and} \quad \tilde{x}_{k,t} = \psi'_k m_t + u_{k,t}, \quad k = 1, \dots, K,$$

respectively.

The quasi log-likelihood function is given by,

$$\ell(r, x; \theta, \Sigma_u) = -\frac{1}{2} \sum_{t=1}^n [\log(2\pi) + \tilde{h}_t + z_t^2 + K \log(2\pi) + \log(|\Sigma|) + u'_t \Sigma^{-1} u_t],$$

where $z_t = z_t(\theta) = (r_t - \mu)/\sqrt{h_t}$ and $u_{k,t}(\theta) = \tilde{x}_{k,t} - \xi_k - \varphi_k \tilde{h}_t(\theta) - \delta'_k b(z_t(\theta))$. We estimate the model's parameters by maximizing the quasi log-likelihood function, $\ell(r, x; \theta, \Sigma)$, with respect to θ and Σ . The log-likelihood function has a convenient structure, so that (partial) maximization with respect to Σ , for a given value of θ , has the simple solution:

$$\hat{\Sigma}(\theta) = \frac{1}{n} \sum_{t=1}^n u_t(\theta) u_t(\theta)',$$

where we have made explicit that u_t depends on θ but, importantly, does not depend on the covariance matrix Σ . We can therefore simplify the maximization problem to $\arg \max_{\theta} \ell(r, x; \theta, \hat{\Sigma}(\theta))$, where

$$\ell(r, x; \theta, \hat{\Sigma}(\theta)) \propto -\frac{1}{2} \sum_{t=1}^n [\log h_t(\theta) + z_t(\theta)^2] - \frac{n}{2} \log \det \hat{\Sigma}(\theta),$$

and where we use the fact that $\sum_{t=1}^n u_t(\theta)' \hat{\Sigma}(\theta)^{-1} u_t(\theta) = \text{tr} \left\{ \sum_{t=1}^n \hat{\Sigma}(\theta)^{-1} u_t(\theta) u_t(\theta)' \right\} = nK$, which does not depend on θ .

In order to compute robust standard errors we need to derive the dynamic properties of the score and hessian. A key component in this dynamics is the derivative of $\log h_{t+1}$ with respect to $\log h_t$, which is stated next.

Lemma 1. *Let $\varphi = (\varphi_1, \dots, \varphi_K)'$ and let D be the matrix whose k -th row is δ'_k , $k = 1, \dots, K$. Then $\partial \log h_{t+1} / \partial \log h_t = A(z_t)$ and $-2\partial \ell_t / \partial \log h_t = B(z_t, u_t)$, where*

$$\begin{aligned} A(z_t) &= (\beta - \gamma' \varphi) + \frac{1}{2} (\gamma' D \dot{b}_{z_t} - \tau' \dot{a}_{z_t}) z_t, \\ B(z_t, u_t) &= (1 - z_t^2) + u'_t \Sigma^{-1} (D \dot{b}_{z_t} z_t - 2\varphi), \end{aligned}$$

with $\dot{a}_{z_t} = \partial a(z_t) / \partial z_t$ and $\dot{b}_{z_t} = \partial b(z_t) / \partial z_t$.

We obtain the following results for $\dot{h}_{\lambda,t} = \frac{\partial \tilde{h}_t}{\partial \lambda}$ and $\dot{h}_{\mu,t} = \frac{\partial \tilde{h}_t}{\partial \mu}$, which we use to simplify our expressions for the score function.

Lemma 2. $\dot{h}_{\lambda,t} = \frac{\partial \tilde{h}_t}{\partial \lambda}$ and $\dot{h}_{\mu,t} = \frac{\partial \tilde{h}_t}{\partial \mu}$ are given from the stochastic recursions:

$$\begin{aligned}\dot{h}_{\lambda,t+1} &= A(z_t)\dot{h}_{\lambda,t} + g_t, \\ \dot{h}_{\mu,t+1} &= A(z_t)\dot{h}_{\mu,t} + (\gamma'D\dot{b}_{z_t} - \tau'\dot{a}_{z_t})h_t^{-\frac{1}{2}},\end{aligned}$$

for $t \geq 1$ with $\dot{h}_{\lambda,1} = \dot{h}_{\mu,1} = 0$.

Next we turn to the score that defines the first order conditions for the quasi maximum likelihood estimators.

Theorem 1. The scores, $\frac{\partial \ell}{\partial \theta} = \sum_{t=1}^n \frac{\partial \ell_t}{\partial \theta}$ with $\theta = (\tilde{h}_1, \mu, \lambda', \psi'_1, \dots, \psi'_K)'$ and $\frac{\partial \ell}{\partial \Sigma^{-1}} = \sum_{t=1}^n \frac{\partial \ell_t}{\partial \Sigma^{-1}}$ are given from

$$\frac{\partial \ell_t}{\partial \theta} = -\frac{1}{2} \begin{pmatrix} B(z_t, u_t) \prod_{s=1}^{t-1} A(z_s) \\ B(z_t, u_t)\dot{h}_{\mu,t} + 2[z_t - u_t'\Sigma^{-1}D\dot{b}_{z_t}]h_t^{-\frac{1}{2}} \\ B(z_t, u_t)\dot{h}_{\lambda,t} \\ -2\Sigma^{-1}u_t \otimes m_t \end{pmatrix} \quad \text{and} \quad \frac{\partial \ell_t}{\partial \Sigma^{-1}} = \frac{1}{2}(\Sigma - u_t u_t'). \quad (1)$$

From Lemma 1 and Theorem 1 it follows that:

Corollary 1. The score function is a martingale difference process, provided that $E(z_t|\mathcal{F}_{t-1}) = 0, E(z_t^2|\mathcal{F}_{t-1}) = 1, E(u_t|z_t, \mathcal{F}_{t-1}) = 0$ and $E(u_t u_t'|\mathcal{F}_{t-1}) = \Sigma$.

The first-order conditions for Σ lead to the close-form expression

$$\hat{\Sigma} = \frac{1}{n} \sum_{t=1}^n \hat{u}_t \hat{u}_t', \quad (2)$$

where $\hat{u}_{k,t} = u_{k,t}(\hat{\theta}) = \tilde{x}_{k,t} - \hat{\xi}_k - \hat{\varphi}_k \tilde{h}_t(\hat{\theta}) - \hat{\delta}_k' b(z_t(\hat{\theta}))$, $k = 1, \dots, K$. This expression reduces the complexity of the optimization problem substantially.

The GARCH equation implies that $\log h_t$ has the stationary MA(∞) representation

$$\log h_t = \beta^j \log h_{t-j} + \sum_{i=0}^{j-1} \beta^j [\tau(z_{t-1-i}) + \gamma' u_{t-1-i}],$$

so that h_t has a stationary representation if $|\beta| < 1$. In the likelihood analysis, however, it is the random variable, $A(z_t) = (\beta - \gamma'\varphi) + \frac{1}{2}(\gamma'D\dot{b}_{z_t} - \tau'\dot{a}_{z_t})z_t$, that shows up as the “autoregressive coefficient” in various expressions. This occurs because the derivative is taken while the observables (r_t, x_t) are held constant, and this phenomenon is well known from the EGARCH model. We note that the effect that h_1 has on the log-likelihood is proportional to $\sum_{t=1}^n \prod_{s=1}^{t-1} A(z_s)$, and that $\prod_{s=1}^{t-1} A(z_s)$ vanishes in

probability (exponentially fast) provided that

$$\mathbb{E} \log |A(z_t)| < 0, \quad (3)$$

because $|\prod_s A(z_s)| = \exp n \left\{ \frac{1}{n} \sum_s \log |A(z_s)| \right\}$ and $\frac{1}{n} \sum_s \log |A(z_s)| \xrightarrow{a.s.} \mathbb{E} \log |A(z_t)|$ by the law of large numbers. By Jensen's inequality we observe that $\mathbb{E}|A(z_t)| < 1$ is a sufficient condition for (3). In our empirical analysis this condition is satisfied in all the models we have estimated. For instance, the estimated model for SPY close-to-close returns with the realized kernel, $x_{RK,t}$, has $\mathbb{E}|A(z_t)| = 0.76$, whereas the estimated model using two realized measures, the realized kernel and the daily range, has $\mathbb{E}|A(z_t)| = 0.835$. (Here the expectation is computed using parameter estimates and a Gaussian specification for z_t).

A drawback of conventional GARCH models is that the asymptotic analysis of estimators and their properties are rather challenging. It took more than two decades to establish several elementary results for some of the simplest models, see Bollerslev and Wooldridge (1992), Lee and Hansen (1994), Lumsdaine (1996), Jensen and Rahbek (2004), Straumann and Mikosch (2006), Kristensen and Rahbek (2005, 2009), and references therein. The asymptotic analysis of Realized EGARCH model is similarly complicated, so it is beyond the scope of this paper to fully establish the asymptotic theory for the estimators. However, based on the theoretical results in this section, including the martingale difference properties stated in Corollary 1, it seems reasonable to make the following assumption about the limit distribution, which we use to compute standard errors in our empirical analysis.

Assumption 1. *The QMLE estimators*

$$\sqrt{n} \begin{pmatrix} \hat{\theta} - \theta \\ \text{vech}(\hat{\Sigma} - \Sigma) \end{pmatrix} \xrightarrow{d} N(\mathcal{I}^{-1} \mathcal{J} \mathcal{I}^{-1}),$$

where \mathcal{J} is the asymptotic variance of the score function and \mathcal{I} is (minus) the limit of Hessian matrix for the log-likelihood function.

In practice we rely on the expression (2) for estimating for $\hat{\Sigma}$, and are mainly concerned with computing standard errors for (elements of) $\hat{\theta}$. Fortunately, \mathcal{I} has a block diagonal structure that simplifies the computation of standard errors for θ .

Theorem 2. *Given the martingale difference conditions stated in Corollary 1 and Assumption 1,*

$$\sqrt{n} (\hat{\theta} - \theta) \xrightarrow{d} N(0, \hat{\mathcal{I}}_{\theta}^{-1} \hat{\mathcal{J}}_{\theta} \hat{\mathcal{I}}_{\theta}^{-1}),$$

In practice we will use this simplification and compute standard errors for θ using $\hat{\mathcal{I}}_{\theta}^{-1} \hat{\mathcal{J}}_{\theta} \hat{\mathcal{I}}_{\theta}^{-1}$, where $\hat{\mathcal{J}}_{\theta}$ will be based on the analytical scores derived in equation (1) and $\hat{\mathcal{I}}_{\theta}$ will be computed from the numerical Hessian matrix of the log-likelihood function.

3.1 Partial Log-Likelihood for Returns

In order to have a measure of fit that can be compared with conventional GARCH models, we define

$$\ell_P(r; \theta) = -\frac{1}{2} \sum_{t=1}^n [\log(2\pi) + \log(h_t) + (r_t - \mu)^2 / h_t],$$

which is the partial log-likelihood function (for the time series of returns). This quantity is the Kullback-Leibler measure associated with the conditional distribution of returns. So this measure is directly comparable to the log-likelihood obtained from conventional GARCH models, such as the GARCH model and the EGARCH model.

4 Empirical Results

In this section we present empirical results using returns and realized measures for 28 stocks and an exchange-traded index fund, SPY, that tracks the S&P 500 index.

The empirical results illustrate the (rather large) benefits of using realized measures in this framework. For instance, the log-likelihood for returns increases substantially when realized measures are included in the modeling, and γ is found to be significant for all realized measures. Models with multiple realized measures lead to better empirical fits than those with a single realized measure. The best combination of realized measures appears to be the realized kernel (see below) in conjunction with the daily range. We explore a number of simplifications of the model structure and find $\varphi = 1$ and $\mu = 0$ to be useful restrictions that, on average, lead to a better out-of-sample fit. Additional results are presented in the next section where we show that the exponential specification proposed in this paper, is superior to the original specification by Hansen et al. (2012). Section 5 will also present theoretical insight about the underlying reasons for this.

4.1 Data and Realized Measures

Our full sample spans the period from January 1, 2002 to August 29, 2008. For out-of-sample analysis, we define an in-sample period: January 1, 2002 to December 31, 2005, leaving January 1, 2006 to August 29, 2008, for out-of-sample analysis. These data were previously analyzed in Hansen, Huang, and Shek (2012), although we include additional realized measures in the present analysis, and we will estimate models that utilize different subsets of eight distinct realized measures. When we estimate a Realized EGARCH model using open-to-close returns we should expect $x_t \approx h_t$, whereas with close-to-close returns we should expect x_t to be smaller than h_t on average.

To avoid outliers that would result from half trading days, we removed the days where high-frequency data spanned less than 90% of the official 6.5 hours between 9:30am and 4:00pm. This removes about three daily observations per year, such as the day after Thanksgiving Day and the days around Christmas.

4.1.1 Realized Measures of Volatility

We use eight realized measures in our empirical analysis. These consist of the Realized Kernel (RK) by Barndorff-Nielsen et al. (2008), the daily range (DR), and six Realized Variances (RV) that differ in terms of the sampling frequency of intraday returns (ranging from fifteen seconds intraday returns to twenty minutes intraday returns).

A relatively simple realized measure of volatility is the sum of squared intraday returns that is known as the realized variance. By dividing some interval of time, $[T_0, T_1]$ say, into n subintervals, $T_0 = t_{0,n} < t_{1,n} < \dots < t_{n,n} = T_1$, we can define the intraday returns, $r_{i,n} = p_{t_{i,n}} - p_{t_{i-1,n}}$. The realized variance is now defined by $RV_t^{(n)} = \sum_{i=1}^n r_{i,n}^2$, and under ideal circumstances the realized variance is consistent for the quadratic variation. (Note that the quadratic variation is an ex-post measure of volatility, as oppose to h_t that is an ex-ante quantity). However, it is well known that market microstructure noise becomes increasingly important as $n \rightarrow \infty$, which makes the RV an unreliable measure of volatility when n is large, see Zhang et al. (2005), Bandi and Russell (2008), and Hansen and Lunde (2006).

The Realized Kernel (RK) by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) is one of several robust measures of volatility, and the variant derived in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011) is robust to general forms of noise. In this paper we adopt the latter variant, which is given by $RK = \sum_{h=-H}^H k\left(\frac{h}{H+1}\right) \gamma_h$, where $k(x)$ is the Parzen kernel and $\gamma_h = \sum_{i=|h|+1}^n r_{i,n} r_{i-h,n}$. The exact computation of this estimator is described in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009).

The daily range is defined by $\text{high}_t - \text{low}_t$ with $\text{high}_t = \max_s p_s$ and $\text{low}_t = \min_s p_s$ where p_t is the logarithmic price and the maximum and minimum is taken over observed prices on the t -th trading day. For the sake of convenience we use the squared daily range,

$$DR_t = (\text{high}_t - \text{low}_t)^2,$$

because this transforms the range-measure to the same scale as h_t . When log-prices follow a Brownian motion with (constant) variance h_t , then $\log DR_t \sim N(0.85 + \log h, 0.34)$ whereas $\log(\text{high}_t - \text{low}_t) \sim N(0.43 + \frac{1}{2} \log h, 0.08)$, see Alizadeh et al. (2002, table 1).

4.2 A Comparison of Realized Measure

First, we compare the performance of Realized EGARCH models that include a single realized measure and compare the results for the eight realized measures. Later we explore the benefits of utilized multiple realized measures simultaneously. Results based on open-to-close returns are presented in Table 2, Panel A, and the analogous results for close-to-close results are presented Panel B.

Point Estimates and Log-Likelihoods (SPY)								Averages Across All Assets							
<i>Panel A: Open-to-Close Returns</i>															
	β	γ	φ	σ_u^2	$\ell_P^{\text{IS}}(r)$	$\ell_P^{\text{OS}}(r)$		β	γ	φ	σ_u^2	$\ell_P^{\text{IS}}(r)$	$\ell_P^{\text{OS}}(r)$		
RK	0.987	0.208	1.093	0.108	-1221.13	-754.04		0.979	0.246	1.078	0.158	-1630.81	-1037.87		
DR	0.997	0.070	1.025	0.260	-1219.09	-757.05		0.987	0.112	1.029	0.310	-1633.16	-1041.29		
RV15s	0.993	0.259	1.226	0.063	-1221.30	-751.84		0.984	0.298	1.077	0.085	-1634.23	-1042.14		
RV2m	0.990	0.204	1.103	0.099	-1220.02	-753.35		0.982	0.257	1.090	0.128	-1631.21	-1038.56		
RV5m	0.988	0.172	1.094	0.134	-1220.88	-755.50		0.980	0.227	1.069	0.171	-1630.80	-1038.32		
RV10m	0.990	0.158	1.050	0.173	-1218.77	-755.63		0.981	0.186	1.058	0.225	-1631.44	-1038.23		
RV15m	0.992	0.114	1.045	0.205	-1222.49	-759.73		0.982	0.157	1.040	0.273	-1632.11	-1038.08		
RV20m	0.993	0.108	1.012	0.227	-1220.86	-756.96		0.983	0.143	1.032	0.309	-1632.28	-1038.59		
<i>Panel B: Close-to-Close Returns</i>															
	β	γ	φ	σ_u^2	$\ell_P^{\text{IS}}(r)$	$\ell_P^{\text{OS}}(r)$		β	γ	φ	σ_u^2	$\ell_P^{\text{IS}}(r)$	$\ell_P^{\text{OS}}(r)$		
RK	0.984	0.229	1.023	0.109	-1301.71	-854.69		0.979	0.255	1.042	0.162	-1744.88	-1130.98		
DR	0.994	0.069	0.966	0.357	-1303.16	-856.32		0.987	0.104	1.006	0.402	-1745.76	-1133.03		
RV15s	0.990	0.283	1.154	0.064	-1299.22	-848.33		0.984	0.307	1.039	0.086	-1749.57	-1133.30		
RV2m	0.986	0.221	1.046	0.100	-1301.11	-853.79		0.982	0.266	1.052	0.130	-1745.79	-1131.22		
RV5m	0.985	0.185	1.032	0.137	-1302.19	-856.33		0.981	0.234	1.035	0.175	-1745.26	-1131.72		
RV10m	0.986	0.164	1.007	0.178	-1302.09	-854.96		0.981	0.193	1.025	0.234	-1745.39	-1131.80		
RV15m	0.988	0.135	0.980	0.215	-1302.90	-860.96		0.982	0.162	1.015	0.285	-1745.95	-1132.03		
RV20m	0.988	0.124	0.957	0.243	-1303.09	-856.89		0.983	0.147	1.005	0.326	-1745.87	-1132.39		

Table 2: Results based on *open-to-close* returns (Panel A) and *close-to-close* returns (Panel B) for eight realized measures of volatility. The left panel presents parameter estimates for SPY returns for four key parameters along with the value of the in-sample and out-of-sample partial log-likelihood function. Each row has the results for one of the eight realized measures in our analysis. The right panel presents the corresponding results for the individual stocks, in the form of average point estimate and average value of the log-likelihood function.

Interestingly, RV15s delivers the best out-of-sample fit for SPY returns but this is not true in general. In fact, the RV15s tends to deliver either the best or the worst out-of-sample performance among different RVs. This finding is likely due to the properties of the microstructure noise that vary across assets. If the microstructure noise is substantial, RV15s tends to provide a poor signal of the underlying integrated volatility. On the other hand for assets where microstructure noise is very small, RV15s may be the most accurate realized measures, resulting in the best out-of-sample fit.

Not surprisingly do we find that the estimates for β are close to 1, see Table 2. This is true uniformly across different stocks for all the different realized measures.

There is an inverse relationship between the coefficient of residual measurement error, γ , and its variance σ_u^2 , (for individual realized measures we write σ_u^2 in place of Σ). This is logical because it implies that the more accurate is a realized measure, the larger is its coefficient in the GARCH equation.

It makes little sense to compare the full log-likelihood $\ell(r, x)$ for different realized measures, because these are log-likelihoods for different data sets. So we will focus on the partial likelihood that is a measure of fit for the return data. Across the 29 assets, the best average out-of-sample log-likelihood is achieved by the realized kernel, RK. In Figure 1 we present the difference between the out-of-sample partial log-likelihood for each of the eight realized measures relative to that of the realized kernel which has the highest average value of the partial log-likelihood. The in-sample analysis for the partial likelihoods has a similar pattern and is not reported to conserve space.

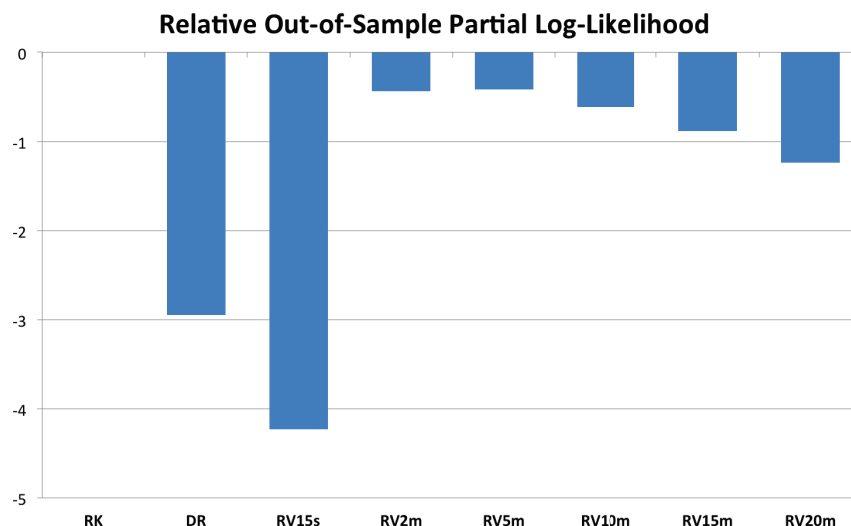


Figure 1: Out-of-sample partial log-likelihood (averaged over assets) for the different realized measures relative to that with the highest average. On average the best empirical fit is achieved with the realized kernel and the worst is that of the noise-prone realized variance based on 15-second returns.

In terms of the out-of-sample log-likelihood the realized kernel (RK) has the best average performance, closely followed by the realized variance based on two-minute sampling. The worst statistical fit

is that of the realized variance based on 15-second sampling followed by the daily range.

There is an inverted U-shape in the score of fit for realized variances as a function of sampling frequency. When the sampling frequency gets higher, the score of fit improves with the peak at around two to five minute frequency but then deteriorate as the sampling frequency increases to 15 seconds. This result is consistent with the literature on market microstructure noise in high frequency data, which has shown that the mean square error (MSE) of the realized variance, as a function of the sampling frequency, has a U-shape. The U-shape arises because the distortions induced by market microstructure noise increase with the sampling frequency. This distortion eventually dominates the statistical gain from increasing the sample size by sampling more frequently.

4.3 Detailed Results for Realized Exponential GARCH with the Realized Kernel

We shall present more detailed results for the Realized EGARCH model based on the RK. The following are the results for SPY open-to-close returns for the full sample period:

$$\begin{aligned} r_t &= \underset{(0.017)}{-0.022} + \sqrt{h_t} z_t \\ \log h_{t+1} &= \underset{(0.005)}{-0.015} + \underset{(0.005)}{0.970} \log h_t - \underset{(0.009)}{0.105} z_t + \underset{(0.005)}{0.051} (z_t^2 - 1) + \underset{(0.024)}{0.272} u_t \\ \log x_{RK,t} &= \underset{(0.042)}{-0.161} + \underset{(0.046)}{1.096} \log h_t - \underset{(0.010)}{0.076} z_t + \underset{(0.006)}{0.073} (z_t^2 - 1) + u_t, \end{aligned}$$

with $\hat{\sigma}_u^2 = \underset{(0.005)}{0.132}$. The numbers in parentheses are the robust standard errors for each of the point estimates.

Estimating the same specification, for SPY close-to-close returns yields:

$$\begin{aligned} r_t &= \underset{(0.019)}{+0.009} + \sqrt{h_t} z_t \\ \log h_{t+1} &= \underset{(0.004)}{-0.008} + \underset{(0.005)}{0.970} \log h_t - \underset{(0.010)}{0.130} z_t + \underset{(0.004)}{0.026} (z_t^2 - 1) + \underset{(0.024)}{0.270} u_t \\ \log x_{RK,t} &= \underset{(0.039)}{-0.399} + \underset{(0.049)}{1.066} \log h_t - \underset{(0.010)}{0.107} z_t + \underset{(0.005)}{0.034} (z_t^2 - 1) + u_t, \end{aligned}$$

with $\hat{\sigma}_u^2 = \underset{(0.005)}{0.134}$.

The estimates are very similar, with the exception of the intercept parameter in the measurement equation, ξ , which is smaller for close-to-close returns than for open-to-close returns. This is to be expected because it is the same realized kernel estimate that is used in both specification, with the implication that $x_{RK,t}$ is a downwards biased measurement of h_t , when the latter is the daily close-to-close volatility. The fact that γ is significant shows that the realized measure (RK) provides valuable information about the variation in volatility, over and above that explained by studentized returns, z_t . We also observe that the coefficients for z_t are smaller (more negative) for close-to-close returns, which suggests a higher degree of asymmetry in the leverage effect. The mean parameter, μ , is estimated to

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	σ_u^2
AA	-0.126	0.040	0.960	0.287	-0.053	0.049	0.020	1.008	-0.043	0.087	0.156
AIG	-0.069	0.015	0.974	0.348	-0.075	0.025	-0.070	0.985	-0.026	0.044	0.196
AXP	0.030	0.007	0.984	0.325	-0.076	0.048	-0.156	1.079	-0.012	0.092	0.173
BA	-0.005	0.015	0.975	0.218	-0.043	0.031	-0.179	1.225	-0.017	0.090	0.157
BAC	0.034	0.001	0.977	0.397	-0.086	0.050	-0.016	0.972	-0.032	0.077	0.170
C	-0.054	0.005	0.982	0.365	-0.080	0.050	0.083	0.974	-0.033	0.089	0.149
CAT	0.002	0.034	0.953	0.271	-0.044	0.040	-0.156	1.109	-0.022	0.088	0.142
CVX	0.015	0.018	0.944	0.272	-0.055	0.046	-0.101	1.293	-0.078	0.073	0.145
DD	-0.015	0.011	0.962	0.294	-0.056	0.042	0.065	1.092	-0.043	0.079	0.161
DIS	0.042	0.011	0.979	0.306	-0.043	0.034	-0.040	1.097	-0.039	0.088	0.166
GE	-0.022	0.002	0.982	0.264	-0.044	0.050	0.000	0.996	0.004	0.077	0.166
GM	-0.178	0.036	0.973	0.230	-0.048	0.068	-0.321	1.050	-0.024	0.118	0.217
HD	-0.012	0.017	0.975	0.280	-0.061	0.042	-0.018	1.030	-0.038	0.090	0.161
IBM	0.072	0.003	0.971	0.307	-0.065	0.044	0.022	0.960	-0.022	0.080	0.145
INTC	-0.053	0.034	0.967	0.330	-0.053	0.053	-0.163	1.074	-0.022	0.075	0.134
JNJ	0.011	-0.008	0.978	0.290	-0.044	0.035	0.135	1.036	0.020	0.099	0.192
JPM	0.006	0.011	0.981	0.350	-0.072	0.052	-0.030	1.013	-0.028	0.086	0.172
KO	0.028	-0.007	0.972	0.356	-0.025	0.035	0.158	0.953	-0.012	0.076	0.151
MCD	0.103	0.012	0.974	0.264	-0.030	0.033	-0.010	0.993	-0.025	0.109	0.200
MMM	-0.019	0.004	0.936	0.326	-0.058	0.027	0.036	1.062	-0.019	0.071	0.166
MRK	-0.006	0.028	0.949	0.239	-0.025	0.012	-0.256	1.340	-0.009	0.070	0.230
MSFT	0.014	0.009	0.971	0.333	-0.054	0.060	0.070	0.920	-0.029	0.081	0.147
PG	0.090	-0.019	0.941	0.319	-0.049	0.041	0.185	1.070	-0.032	0.078	0.168
T	-0.035	0.025	0.960	0.346	-0.070	0.073	-0.001	0.908	-0.037	0.098	0.204
UTX	-0.027	0.012	0.958	0.319	-0.082	0.048	0.028	0.936	-0.015	0.099	0.154
VZ	-0.043	0.008	0.980	0.279	-0.046	0.042	0.056	1.024	-0.033	0.083	0.180
WMT	-0.034	0.003	0.975	0.270	-0.044	0.047	0.129	1.074	-0.016	0.091	0.152
XOM	0.047	0.015	0.953	0.248	-0.060	0.043	-0.108	1.278	-0.075	0.075	0.135
SPY	-0.022	-0.015	0.969	0.272	-0.105	0.051	-0.161	1.096	-0.076	0.073	0.132
Average	-0.008	0.011	0.967	0.300	-0.057	0.044	-0.028	1.057	-0.029	0.084	0.166

Table 3: Estimates for the Realized EGARCH model based on the realized kernel (RK) for open-to-close returns.

be negative for open-to-close returns and positive for close-to-close returns for this sample period, albeit both estimates are insignificant.

The point estimates for each of the DIJA stocks are presented in Table 3 (open-to-close returns) and Table 4 (close-to-close returns). The last row has the average estimates across all assets, and it is interesting to observe how similar the point estimates are across assets. Not surprisingly, do we find volatility to be highly persistent, which is evident from the estimates of β , that are close to 1 in all cases. Moreover, the tables show that γ is typically estimated to be about 0.30, for both open-to-close returns and close-to-close returns. This parameter may be compared to α in a conventional GARCH model, which measures the coefficient associated with squared returns. The fact that γ is estimated to be several times larger than the typical value for α (about 0.05) reflects the fact that the realized measures provide better information about future volatility than does the squared return.

Stocks	μ	ω	β	γ	τ_1	τ_2	ξ	φ	δ_1	δ_1	σ_u^2
AA	0.013	0.055	0.959	0.267	-0.045	0.030	-0.507	1.130	-0.058	0.077	0.154
AIG	-0.027	0.020	0.973	0.360	-0.093	0.046	-0.217	0.924	-0.052	0.066	0.188
AXP	0.016	0.012	0.984	0.310	-0.101	0.040	-0.403	1.057	-0.037	0.067	0.176
BA	0.061	0.021	0.976	0.227	-0.059	0.032	-0.432	1.114	-0.030	0.087	0.153
BAC	0.010	0.007	0.975	0.366	-0.104	0.047	-0.217	0.990	-0.060	0.068	0.170
C	-0.006	0.009	0.983	0.347	-0.089	0.033	-0.135	0.969	-0.041	0.055	0.158
CAT	0.080	0.045	0.956	0.266	-0.046	0.022	-0.565	1.183	-0.042	0.068	0.144
CVX	0.068	0.030	0.946	0.256	-0.059	0.043	-0.406	1.303	-0.102	0.057	0.143
DD	0.009	0.021	0.963	0.282	-0.061	0.022	-0.243	1.130	-0.049	0.058	0.161
DIS	0.016	0.015	0.980	0.299	-0.058	0.025	-0.281	1.037	-0.050	0.062	0.169
GE	0.000	0.007	0.984	0.284	-0.060	0.039	-0.244	0.971	-0.012	0.059	0.173
GM	-0.027	0.047	0.970	0.224	-0.049	0.060	-0.774	1.158	-0.035	0.110	0.219
HD	-0.025	0.021	0.978	0.281	-0.060	0.036	-0.256	1.007	-0.029	0.068	0.168
IBM	-0.001	0.012	0.973	0.355	-0.084	0.027	-0.246	0.871	-0.036	0.053	0.150
INTC	0.004	0.038	0.971	0.337	-0.055	0.036	-0.427	1.048	-0.025	0.046	0.140
JNJ	0.016	-0.002	0.977	0.225	-0.056	0.039	-0.118	1.158	0.008	0.083	0.194
JPM	0.012	0.014	0.983	0.317	-0.089	0.051	-0.235	0.987	-0.054	0.066	0.175
KO	0.013	0.000	0.971	0.393	-0.040	0.025	-0.086	0.888	-0.021	0.056	0.154
MCD	0.091	0.023	0.968	0.248	-0.046	0.032	-0.314	1.046	-0.068	0.101	0.196
MMM	0.011	0.023	0.942	0.288	-0.063	0.009	-0.365	1.158	-0.033	0.042	0.169
MRK	0.017	0.042	0.946	0.205	-0.033	0.018	-0.708	1.497	-0.041	0.077	0.219
MSFT	0.010	0.016	0.977	0.345	-0.051	0.030	-0.246	0.928	-0.025	0.045	0.154
PG	0.019	-0.004	0.936	0.302	-0.057	0.029	-0.093	1.161	-0.043	0.067	0.164
T	0.010	0.025	0.967	0.356	-0.065	0.053	-0.106	0.913	-0.043	0.062	0.222
UTX	0.018	0.024	0.958	0.340	-0.102	0.050	-0.188	0.885	-0.033	0.090	0.155
VZ	-0.002	0.013	0.980	0.267	-0.061	0.039	-0.171	1.006	-0.048	0.064	0.185
WMT	-0.020	0.012	0.974	0.265	-0.043	0.037	-0.273	1.121	-0.012	0.067	0.158
XOM	0.055	0.025	0.956	0.239	-0.071	0.039	-0.401	1.256	-0.102	0.058	0.133
SPY	0.009	-0.008	0.970	0.270	-0.130	0.026	-0.399	1.066	-0.107	0.034	0.134
Average	0.016	0.019	0.968	0.294	-0.067	0.035	-0.312	1.068	-0.044	0.066	0.168

Table 4: Estimates for the Realized EGARCH model based on the realized kernel (RK) for close-to-close returns.

4.4 Simplifying the Structure Through Parameter Restrictions

We seek ways to simplify the model by imposing parameter restrictions that are not at odds with the data. There are several advantages of imposing restrictions. For instance, it can ease the interpretation of the model and can make the estimation of the remaining parameters more efficient. In this section we will explore the validity of the following two restrictions: $\mu = 0$ and $\varphi = 1$.

4.4.1 Imposing the Restriction: $\mu = 0$

We first check whether the restriction $\mu = 0$ in the mean equation is justified by the empirical results. Table 5 presents the estimated μ and the differenced log-likelihoods between Realized EGARCH model with and without μ . We can see that leaving μ unrestricted will, on average, not improve the in-sample fit significantly and, in fact, imposing $\mu = 0$ leads to a better average out-of-sample fit. For this reason we shall impose $\mu = 0$ in the remaining empirical analysis. These results are based on estimates where the realized kernel was used as the realized measure. The (unreported) results for the other realized measures are very similar.

Stocks	μ	$\ell(r, x)^{IS}$	$\ell(r)_P^{IS}$	$\ell(r, x)^{OS}$	$\ell(r)_P^{OS}$
AA	-0.14	-4.76	-4.30	-0.72	-0.47
AIG	-0.04	-0.44	-0.28	-2.00	-1.93
AXP	0.03	-0.45	-0.53	-0.36	0.07
BA	0.04	-0.52	-0.72	1.56	1.95
BAC	0.05	-1.92	-1.42	0.76	2.29
C	-0.06	-1.69	-2.24	-0.65	-1.37
CAT	0.00	0.00	0.00	0.00	0.00
CVX	0.01	-0.09	-0.19	-0.07	-0.11
DD	-0.04	-0.61	-0.66	0.83	0.70
DIS	-0.01	-0.02	-0.05	0.54	0.44
GE	-0.03	-0.50	-0.60	0.14	0.19
GM	-0.20	-9.69	-10.27	1.86	0.16
HD	-0.04	-0.39	-0.52	0.62	0.42
IBM	0.00	0.00	0.00	0.04	0.04
INTC	-0.09	-1.55	-1.76	1.02	1.13
JNJ	0.00	0.00	-0.01	0.06	0.08
JPM	-0.04	-0.55	-0.34	1.80	1.30
KO	0.03	-0.49	-0.56	-0.61	-0.30
MCD	0.11	-3.25	-2.78	-3.11	-2.34
MMM	-0.03	-0.37	-0.40	0.12	-0.06
MRK	-0.05	-0.60	-0.65	0.96	1.06
MSFT	-0.04	-0.83	-0.57	3.09	2.56
PG	0.07	-3.43	-3.70	-5.94	-5.75
T	-0.05	-0.64	-0.76	0.03	-0.24
UTX	-0.02	-0.11	-0.03	-0.32	-0.34
VZ	-0.09	-3.15	-3.68	3.19	2.86
WMT	-0.07	-2.51	-2.45	2.39	2.47
XOM	0.05	-1.00	-0.98	-0.56	-0.92
SPY	-0.04	-1.21	-1.05	1.07	0.44
Average	-0.02	-1.41	-1.43	0.20	0.15

Table 5: Unrestricted point estimates for μ , and the impact on the the joint and partial log-likelihoods from imposing $\mu = 0$.

4.4.2 Imposing the Restriction: $\varphi = 1$

There are theoretical reasons to expect that the realized measures are proportional to h_t . Since we operate with logarithmically transformed quantities it is therefore reasonable to expect that $\varphi = 1$. Imposing this constraint makes it easier to interpret certain features of the model, so we examine the validity of this restriction. We estimate the Realized EGARCH model with each of the eight realized measures and evaluate the effects on the joint and partial log-likelihood functions by imposing the constraint $\varphi = 1$. The results are presented in Figure 2 where the left panel displays the in-sample and out-of-sample effects on the joint log-likelihood, and the results for the partial log-likelihood are presented in the right panel. The numbers reported in the left panel are $\ell(r, x; \tilde{\theta}, \tilde{\Sigma}) - \ell(r, x; \hat{\theta}, \hat{\Sigma})$ (average over the 29 assets) where $\tilde{\theta}$ and $\tilde{\Sigma}$ are the point estimates from the restricted model (where $\varphi = 1$ is imposed), while $\hat{\theta}$ and $\hat{\Sigma}$ are the point estimates from the unrestricted model. The right panel presents the equivalent statistics for the partial log-likelihood function.

Because the parameters are estimated by maximizing the joint likelihood (in-sample), it is no surprise that the in-sample log-likelihood decreases in value when $\varphi = 1$ is imposed. This is not necessarily the case for the partial log-likelihood and, in fact, the daily range provides an example where the in-sample partial log-likelihood increases by imposing $\varphi = 1$. (The implication is that the marginal in-sample log-likelihood for the realized measures decreases more than the joint log-likelihood). The interesting result in the left panel is that the restriction improves the out-of-sample fit on average. We note that out-of-sample improvement is about 2, for both joint and partial likelihoods. This implies that the gains from imposing $\varphi = 1$ are primarily driven by improved out-of-sample fit of returns, and less so by out-of-sample fit of the marginal model for the realized measure. This is obviously desirable if the key objective is a better out-of-sample model for returns. We conclude that $\varphi = 1$ is a reasonable restriction to impose in this framework.

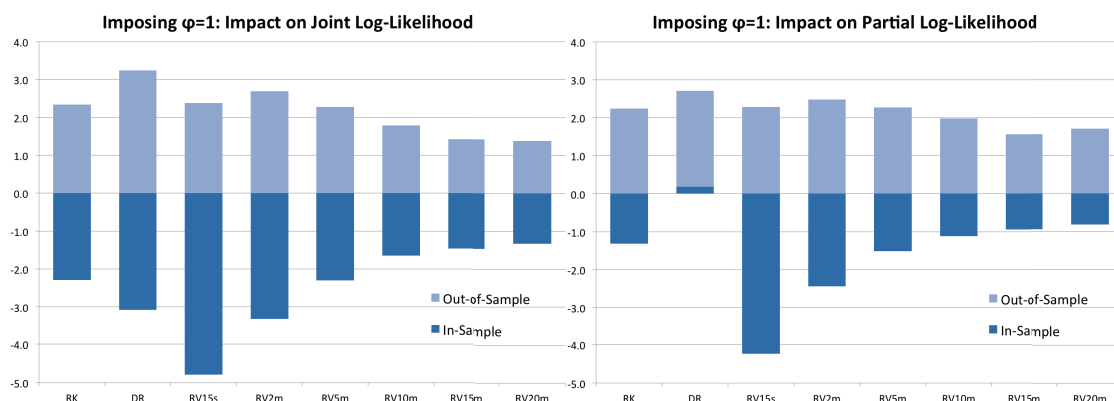


Figure 2: Gains in the joint and partial log-likelihoods by imposing $\varphi = 1$. While restrictions will reduce the value of the in-sample likelihood we note that imposing $\varphi = 1$ leads to improvements in the out-of-sample log-likelihood. Improvements are observed for both the joint and the partial log-likelihoods out-of-sample. The reported values are the average gains across all assets.

4.5 Realized Exponential GARCH with Multiple Realized Measures

In this section, we present empirical results for Realized EGARCH models using multiple realized measures. Models using different combinations of realized measures have been estimated for $M = 2, 3, 4$.

Seven Realized EGARCH models are estimated with different pairs realized measures. We estimate two models with three realized measures, where the labels RK&DR&RV15s and RK&DR&RV5m, identify which realized measures are included in the model. We estimate two models with four realized measures: RK&RV2m&RV5m&RV20m and RK&DR&RV5m&RV20m.

From the models with two realized measures it is interesting to note that including the realized range tends to improve the out-of-sample likelihood. This suggests that the daily range, DR, contains supplementary information to that provided by the realized kernel and the realized variances.

Specification	Partial log-likelihood: ℓ_r (average over assets)		Change in ℓ_r relative to best specification	
	IS	OS	IS	OS
EGARCH	-1637.67	-1055.17	-6.84	-19.75
RK	-1632.13	-1035.64	-1.31	-0.27
DR	-1632.98	-1038.59	-2.16	-3.22
RV15s	-1638.46	-1039.86	-7.64	-4.49
RV2m	-1633.65	-1036.07	-2.82	-0.70
RV5m	-1632.31	-1036.05	-1.49	-0.68
RV10m	-1632.56	-1036.25	-1.74	-0.88
RV15m	-1633.05	-1036.52	-2.23	-1.15
RV20m	-1633.09	-1036.88	-2.27	-1.51
RK&DR	-1631.58	-1035.37	-0.76	0
RK&RV15s	-1640.10	-1040.58	-9.28	-5.21
RK&RV5m	-1631.71	-1035.50	-0.89	-0.13
RV15s&RV5m	-1639.27	-1040.08	-8.44	-4.71
RV5m&RV20m	-1632.03	-1035.84	-1.21	-0.47
DR&RV5m	-1631.49	-1035.71	-0.67	-0.34
DR&RV20m	-1631.88	-1036.62	-1.06	-1.25
RK&DR&RV15s	-1638.96	-1040.28	-8.13	-4.91
RK&DR&RV5m	-1631.12	-1035.38	-0.30	-0.01
RK&RV2m&RV5m&RV20m	-1633.13	-1040.20	-2.31	-0.54
RK&DR&RV5m&RV20m	-1630.82	-1035.38	0	-0.01

Table 6: In-sample and out-of-sample partial log-likelihoods for various specifications measures.

Models that include RV15s typically yield a worse out-of-sample fit whereas most other specifications produce similar log-likelihood values. Comparing the out-of-sample partial log-likelihoods with those of the univariate model, we find incorporating multiple realized measures does improve the value of the likelihood. The results with $M = 3$ realized measures are similar to the case of $M = 2$, with the exception being the case where we include the noise-sensitive realized variance, RV15s.

In Table 6 we report the average value of the partial log-likelihood function for various specifications, where the average is taken over assets. The first and second columns report the in-sample and out-of-sample results respectively. The last two columns report the values of the log likelihood relative to that obtained for the specification with the highest average value. In-sample, the largest average partial log-likelihood is (not surprisingly) achieved by a specification with four realized measures. (RK, DR, RV5m, and RV20m) whereas the simpler specification with two realized measures, RK & DR, has the best out-of-sample fit on average. This suggests the daily range contributes additional information about volatility, even after including the realized kernel or realized variance. Several specifications produce very similar in-sample and out-of-sample log-likelihoods. Specifications that include both the realized kernel and the daily range deliver good overall performance, unless these are used in conjunction with the noise-prone realized variance, RV15s. Incorporating a realized variance sampled at frequencies 5-minute (or slower) does not hurt the empirical fit, whereas the realized variance based on 15-second sampling results in the worst performance across the Realized EGARCH specifications.

5 Relation to Earlier Specification in Hansen, Huang and Shek (2012)

In this section we compare the Realized Exponential GARCH specification (introduced in this paper) with the original specification in Hansen et al. (2012). The latter was formulated for a single realized measure, so we focus on this special case in our comparison. The logarithmic specification in Hansen et al. (2012) takes the form:

$$\begin{aligned}\log h_t &= \tilde{\omega} + \tilde{\beta} \log h_{t-1} + \gamma \log x_{t-1}, \\ \log x_t &= \xi + \varphi \log h_t + \delta(z_t) + u_t.\end{aligned}$$

Compared to the Realized EGARCH specification we note that the GARCH equation has the realized measure, x_{t-1} , instead of u_{t-1} , and lacks a leverage function, $\tau(z_{t-1})$. By substitution we have

$$\log h_t = \omega + \beta h_{t-1} + \gamma \delta(z_{t-1}) + \gamma u_{t-1}, \quad (4)$$

where $\omega = \tilde{\omega} + \gamma \xi$ and $\beta = \tilde{\beta} + \gamma \varphi$. It is now evident that this model is nested in the Realized EGARCH model, as the original specification arises by imposing two restrictions: 1) Proportionality of the two leverage functions, $\tau(z) = \gamma \delta(z)$, and 2) that their relative magnitude is exactly γ (the coefficient for u_{t-1}). Our empirical analysis will highlight the benefits of relaxing these constraints, and we shall provide some theoretical insight about this below.

5.1 Interpreting the Generalized Structure

Before we present the empirical comparison of the two specifications, we will motivate the need for the more flexible structure of the Realized EGARCH specification.

The realized measures, x_t , are estimates of the quadratic variation, which we can denote by y_t . For instance, the realized kernel is consistent for y_t as the number of intraday returns, n , increases. In fact, the variant of the realized kernel used in this paper is such that $x_t - y_t = O_p(n^{-1/5})$ under suitable conditions, see Barndorff-Nielsen et al. (2011). More generally, we can view each of the realized measures as noisy measure of y_t with varying degrees of accuracy. This translates into $\log x_t$ being a noisy measure of $\log y_t$, with a bias that is influenced by the sampling error of the realized measure. So we introduce $\eta_t = \log x_t - \log y_t$ and $\zeta_t = \log y_t - \log h_t$ and label these as *estimation error* and *volatility shock* respectively. While it is plausible that the volatility shock influences the dynamics of volatility, there is little reason to expect that the estimation error has any impact on future volatility. This follows from the fact that η_t is specific to the realized measure and simply reflects our inability to perfectly estimate y_t from a finite number of observations.

From the measurement equation, where we have imposed $\varphi = 1$, we have

$$\xi + \delta(z_t) + u_t = \eta_t + \zeta_t.$$

This enables us to interpret ξ and relate $\delta(z_t)$ and u_t to the estimation error and the volatility shock. First we note that ξ is tied to the sampling error of the realized measure. For a realized measure that is unbiased for y_t , it follows by Jensen's inequality that a larger sampling error decreases $E\eta_t$. Thus if we compare two unbiased realized measures we should expect the more accurate one to have the larger (less negative) value of ξ . Second, since η_t is tied to sampling error that (in the limit for some of the realized measures) is independent of the observed processes, it follows that the leverage function, $\tau(z_t)$, is linked to the volatility shock ζ_t , albeit there will be residual randomness in ζ_t that cannot be explained by the studentized return, z_t , alone. Consequently, the residual measurement shock u_t will be a mixture of the estimation error, η_t , and the residual randomness $\zeta_t - \delta(z_t)$. For this reason we should expect $\delta(z_t)$ to be more important in describing the dynamic variation in volatility than u_t . A limitation of the original Realized GARCH specification is that it implicitly imposes $\delta(z_t)$ and u_t to have the same coefficient in the GARCH equation, see (4).

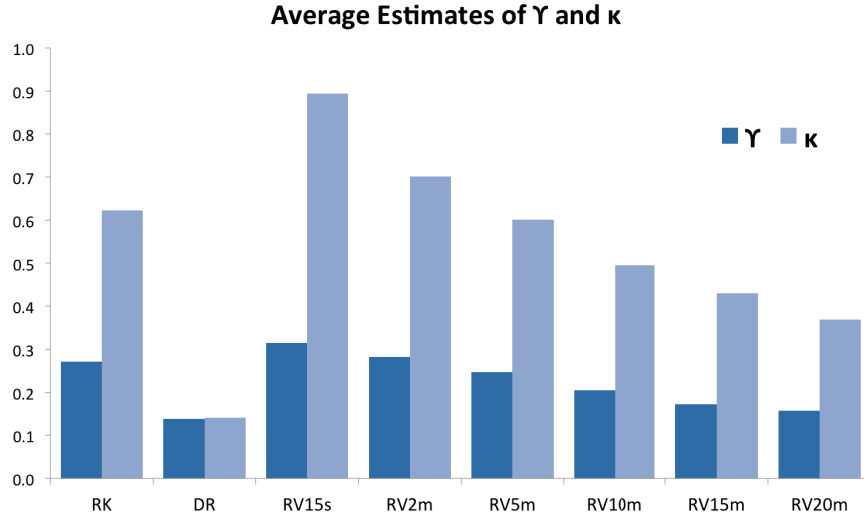


Figure 3: Averaged estimates for γ and κ

5.2 Empirical Comparisons to Realized GARCH

Before comparing the Realized GARCH and Realized EGARCH models we estimate a hybrid model where we relax the constraint that $\delta(z_t)$ and u_t have the same coefficient in the GARCH equation. This is achieved by imposing $\tau(z) = \kappa\delta(z)$ in the GARCH equation of the EGARCH model, where κ is a free parameter. The Realized GARCH model corresponds to the case $\kappa = \gamma$. Figure 3 presents average estimates of γ and κ for the various realized measures, where the averages were taken across assets.

Across stocks we typically find the value of γ to be much smaller than κ (γ is typically estimated to be about 60% smaller than κ , which is consistent with our interpretations of $\delta(z_t)$ and u_t and their relations to estimation error and volatility shocks. Detailed results for the individual stocks are presented in the appendix.

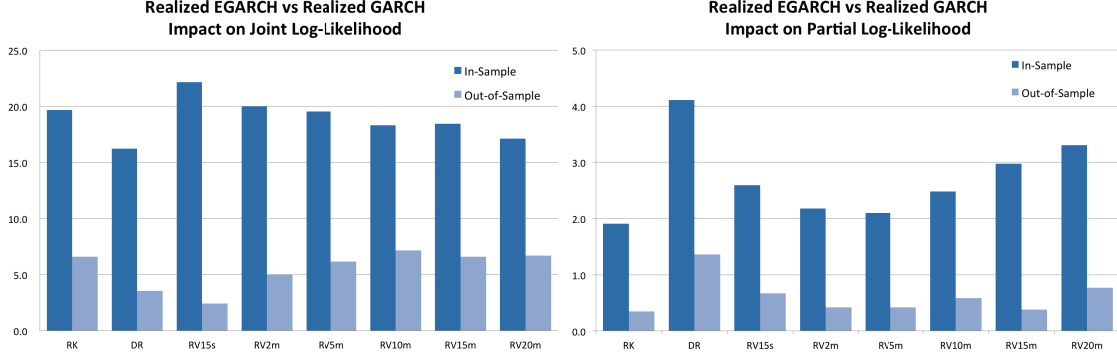


Figure 4: Average impact on the joint and partial log-likelihoods from Realized GARCH to Realized EGARCH

Naturally, the Realized EGARCH model is more flexible than merely allowing $\gamma \neq \kappa$ and Figure 4 presents likelihood ratio statistics (averaged across assets) from the comparison of the Realized EGARCH model and the nested Realized GARCH model. In-sample and out-of-sample statistics are presented for the joint likelihood in the left panel and the corresponding results for the partial log-likelihood are presented in the right panel.

We see both positive in-sample and out-of-sample gains in both joint and partial log-likelihoods. A more flexible specification will always result in a better log-likelihood (the quantity being maximized). What is impressive, is the fact that the out-of sample log-likelihood is also improved in all cases. This strongly suggests that the new specification is superior to that in Hansen et al. (2012). The improvements are also seen in terms of the partial log-likelihoods.

6 Conclusion

We have introduced a new variant of the Realized GARCH model, that is characterized by two innovations. We included an explicit leverage term in the GARCH equation and we allowed for the inclusion of multiple realized measures in the model. The advantages of the new structure were documented in the form of better empirical fit in the time series we have analyzed. In our empirical analysis we also explored two simplifications ($\mu = 0$ and $\varphi = 1$) that simplify the interpretation of the model and facilitate a simpler and more accurate estimation of the model.

We included and compared eight realized measures of volatility in the analysis, a realized kernel (RK), the daily range (DR), and six realized variances, that were computed for various sampling frequencies of intraday returns, ranging from 15 second returns to 20 minute returns. Any of these realized measures

contain useful information for the modeling of volatility. The daily range adds the least in terms of empirical fit, and the realized kernel adding the most. For the realized variances we find, not surprisingly, that the performance improves as the sampling frequency increases, except for the highest sampling frequency where market microstructure noise becomes a dominating factor. The realized variance based on 15 second intraday returns, RV15s, was (in some regards) found to be the best single measure of volatility in our analysis of the SPY returns, albeit the effects of market microstructure noise were evident from the point estimates. However, serious problems arose for RV15s when this measure was used in our analysis of the individual returns series. The core of the problems is that the dynamic properties of RV15s are influenced by the noise, causing the RV15s to “highjack” the latent volatility variable, h_t , to (partly) track these noise features instead of the intended purpose, which is to track the conditional volatility of returns.

The extension to multiple realized measures of volatility was found to be beneficial. Not only does multiple realized measure lead to substantial improvements of the empirical fit in-sample, it also improves the out-of-sample fit. The latter implies that the population benefits from adding multiple measures outweighs the drawback from having to estimate additional parameters in these models. In terms of the average out-of-sample fit of the log-likelihood for returns, the best combination of realized measures was one with two realized measures – the realized kernel paired with the daily range. While the daily range, in isolation, yields the smallest empirical gains of all realized measures, it does contain valuable information that is orthogonal to that of other realized measures. Given the construction of the realized measures, it is perhaps, not surprising that the daily range, albeit relatively noisy, does capture information that is distinct from that contained in the other realized measures.

Realized measures have proven to be very valuable in GARCH modeling. When estimating a standard GARCH model, the lagged squared returns is typically estimated to have a coefficient around 5% which causes GARCH models to be slow at adjusting the level of volatility. Put simply, it takes many consecutive large squared returns for a GARCH model to realize that volatility has jumped to a new higher level. Including a realized measure in the GARCH equation will typically lead to its estimated coefficient to be estimated to about 30%-60%, and the inclusion of the realized measures will often cause the squared return to be insignificant. The larger coefficient associated with the realized measure, makes the model far more adaptive to sudden changes in volatility, which has obvious benefits. For instance, a realized GARCH model fares far better during the financial crises than does a conventional GARCH model. Despite these benefits it is important to be aware of a potential drawback of the coefficient, γ , being relatively large. The larger coefficient in the GARCH equation implies that an outlier in the realized measure will cause more havoc to h_t . This problem is less pronounced in conventional GARCH models because α is small. (Naturally, α is small because squared returns are noisy measures of volatility). The key message we want to make is the following: A larger coefficient in the GARCH equation requires a higher degree of responsibility in terms of including well behaved realized measures of volatility. In our empirical analysis we found that additional data monitoring is required once realized

measures are included in the model. For instance, we chose to exclude “half” trading days (mostly days around Christmas and Thanksgiving) from the analysis, because the realized measures from such days are outliers. Not only because the period with high-frequency data is shorter, but also because such days tend to be quiet day with relatively low levels of volatility. Conventional GARCH models do not require the same degree of careful monitoring, because an outlier in returns, has a smaller impact on the model-implied volatility.

Finally, the model proposed in this paper included a single lag of h_t and u_t in the GARCH equation. This framework is easy to extend to include multiple lags of h_t and u_t , say p and q lags, respectively. It would be natural to label this model as the RealEGARCH(p,q) model, and the likelihood analysis in Section 3 can be adapted to cover this case at the expense of a more complex exposition.

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Appendix of Proofs

Proof of Lemma 1. From $z_t = (r_t - \mu)e^{-\tilde{h}_t/2}$ we have that

$$\dot{z}_t = \partial z_t / \partial \tilde{h}_t = -\frac{1}{2}z_t. \quad (\text{A.1})$$

Similarly for $u_{k,t} = \tilde{x}_t - \varphi_k \tilde{h}_t - \delta'_k b(z_t)$ we find that $\dot{u}_{k,t} = \partial u_{k,t} / \partial \tilde{h}_t = -\varphi_k - \delta'_k \dot{b}_t \dot{z}_t = \frac{1}{2} \delta'_k \dot{b}_t z_t - \varphi_k$, so that $\dot{u}_t = \partial u_t / \partial \tilde{h}_t$ (the column vector whose k -th element is $\dot{u}_{k,t}$) can be expressed as

$$\dot{u}_t = \frac{1}{2} D \dot{b}_t z_t - \varphi. \quad (\text{A.2})$$

Now recall, $\tilde{h}_{t+1} = g'_t \lambda$, where $g'_t = (1, \tilde{h}_t, a'_t, u'_t)$. Thus the object we seek is given by

$$(\partial g'_t / \partial \tilde{h}_t) \lambda = (0, 1, \dot{z}_t \dot{a}'_t, \dot{u}'_t) \lambda = \beta + \dot{z}_t \dot{a}'_t \tau + (\frac{1}{2} D \dot{b}_t z_t - \varphi)' \gamma = A(z_t),$$

For the second result, $-2 \log \ell_t / \partial \tilde{h}_t = [\tilde{h}_t + z_t^2 + K \log(2\pi) + \log |\Sigma| + u'_t \Sigma^{-1} u_t] / \partial \tilde{h}_t$, we note that $\partial z_t^2 / \partial \tilde{h}_t = -z_t^2$ using (A.1), and the result now follows by combining $\partial u'_t \Sigma^{-1} u_t / \partial u'_t = 2u'_t \Sigma^{-1}$ and (A.2). \square

Proof of Lemma 2. First we note that

$$z_t = (r_t - \mu)e^{-\tilde{h}_t/2} \quad \text{and} \quad u_{k,t} = \tilde{x}_t - \varphi_k \tilde{h}_t - \delta'_k b(z_t),$$

only depend on λ through \tilde{h}_t , (the latter directly and indirectly via z_t). Consequently $g'_t = (1, \tilde{h}_t, a'_t, u'_t)$ only depends on λ through \tilde{h}_t so that $\partial \tilde{h}_{t+1} / \partial \lambda =$

$$\frac{\partial \tilde{h}_{t+1}}{\partial \lambda} = \lambda' \frac{\partial g_t}{\partial \tilde{h}_t} \frac{\partial \tilde{h}_t}{\partial \lambda} + g_t = A(z_t) \dot{h}_{\lambda,t} + g_t.$$

The second result is derived similarly, albeit with some additional terms because z_t (and hence u_t)

depends on μ through another channel than \tilde{h}_t . Specifically,

$$\dot{z}_{\mu,t} = \frac{\partial z_t}{\partial \mu} = -\frac{1}{2}z_t \dot{h}_{\mu,t} - h_t^{-\frac{1}{2}},$$

which has implications for $\dot{u}_{\mu,t} = \partial u_t / \partial \mu$ so that

$$\dot{u}_{\mu,t} = -\varphi \dot{h}_{\mu,t} - D\dot{b}_t \dot{z}_{\mu,t} = (-\varphi + \frac{1}{2}D\dot{b}_t z_t) \dot{h}_{\mu,t} + D\dot{b}_t h_t^{-\frac{1}{2}}$$

so that

$$\begin{aligned} \dot{h}_{\mu,t+1} &= \beta \dot{h}_{\mu,t} + \gamma' \dot{u}_{\mu,t} + \tau' \dot{a}_t \dot{z}_{\mu,t} \\ &= [\beta - \gamma' \varphi + \frac{1}{2}(\gamma' D\dot{b}_t - \tau' \dot{a}_t) z_t] \dot{h}_{\mu,t} + (\gamma' D\dot{b}_t - \tau' \dot{a}_t) h_t^{-\frac{1}{2}} \end{aligned}$$

and the result follows. \square

Proof of Theorem 1. From Lemma 1 we have $\frac{\partial \tilde{h}_{t+i}}{\partial \tilde{h}_t} = \prod_{i=0}^{j-1} A(z_{t+i})$, and the first result follows by the chain rule. For μ we note that μ influences ℓ_t through \tilde{h}_t and through its direct impact on z_t which is the second term of $\dot{z}_{\mu,t} = \partial z_t / \partial \mu = -\frac{1}{2}z_t \dot{h}_{\mu,t} - h_t^{-\frac{1}{2}}$ so that

$$\begin{aligned} -2 \frac{\partial \ell_t}{\partial \mu} &= B(z_t, u_t) \dot{h}_{\mu,t} + (-2\partial \ell_t / \partial z_t)(-h_t^{-\frac{1}{2}}) \\ &= B(z_t, u_t) \dot{h}_{\mu,t} + (2z_t + 2u_t' \Sigma^{-1} \partial u_t / \partial z_t)(-h_t^{-\frac{1}{2}}) \\ &= B(z_t, u_t) \dot{h}_{\mu,t} + 2[z_t + u_t' \Sigma^{-1}(-D\dot{b}_{z_t})] h_t^{-\frac{1}{2}}, \end{aligned}$$

which establishes the second element of $\frac{\partial \ell_t}{\partial \theta}$. Next, λ only impacts ℓ_t through \tilde{h}_t so the third element of $\frac{\partial \ell_t}{\partial \theta}$ follows by combining the results in Lemmas 1 and 2. Next consider the derivatives with respect to ψ_k , which only affects ℓ_t through $u_{k,t}$. Recall $-2\partial \ell_t / \partial u_t = \partial(u_t' \Sigma^{-1} u_t) / \partial u_t = 2\Sigma^{-1} u_t$ so that $\partial(u_t' \Sigma^{-1} u_t) / \partial u_{k,t} = 2e_k' \Sigma^{-1} u_t$ where e_k is the k -th unit vector. Since $\partial u_{k,t} / \partial \psi_k = -m_t$ we have

$$-2\partial \ell_t / \partial \psi_k = 2e_k' \Sigma^{-1} u_t (-m_t) = -2(e_k' \Sigma^{-1} u_t) m_t,$$

By using the fact that for vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ we have that

$$\begin{pmatrix} e_1' ab \\ \vdots \\ e_k' ab \\ \vdots \\ e_n' ab \end{pmatrix} = \begin{pmatrix} a_1 b \\ \vdots \\ a_k b \\ \vdots \\ a_n b \end{pmatrix} = a \otimes b,$$

It follows that $-2\partial \ell_t / \partial (\psi_1', \dots, \psi_K')' = (\Sigma^{-1} u_t) \otimes m_t$. \square

Proof of Theorem 2. We first show the information matrix \mathcal{I} is block diagonal. For each $i, j, k = 1, \dots, K$, we have

$$\begin{aligned}\frac{\partial^2 \ell_t}{\partial(\Sigma^{-1})_{ij} \partial \lambda} &= -\frac{1}{2}(u_{i,t} \dot{u}_{j,t} + u_{j,t} \dot{u}_{i,t}) \dot{h}_t \\ \frac{\partial^2 \ell_t}{\partial(\Sigma^{-1})_{ij} \partial \psi_k} &= -\frac{1}{2}(u_{i,t} m_{k,j,t} + u_{j,t} m_{k,i,t})\end{aligned}$$

where $m_{k,i,t} = \frac{\partial u_{i,t}}{\partial \psi_k} = m_{k,t}$ when $k = i$ and 0 otherwise. Hence, we have

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2 \ell(r, x; \theta, \Sigma^{-1})}{\partial(\Sigma^{-1})_{ij} \partial \lambda}\right] &= \mathbb{E}\left[\sum_{t=1}^n -\frac{1}{2}(u_{i,t} \dot{u}_{j,t} + u_{j,t} \dot{u}_{i,t}) \dot{h}_t\right] = 0, \\ \mathbb{E}\left[\frac{\partial^2 \ell(r, x; \theta, \Sigma^{-1})}{\partial(\Sigma^{-1})_{ij} \partial \psi_k}\right] &= \mathbb{E}\left[\sum_{t=1}^n -\frac{1}{2}(u_{i,t} m_{k,j,t} + u_{j,t} m_{k,i,t})\right] = 0.\end{aligned}$$

Since all cross terms are zero the information matrix \mathcal{I} is block triangular, so that

$$\text{avar}(\hat{\theta}, \text{vech} \hat{\Sigma}) = \begin{pmatrix} \mathcal{I}_\theta^{-1} & 0 \\ 0 & \mathcal{I}_\Sigma^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{J}_\theta & \mathcal{J}_{\theta\Sigma} \\ \mathcal{J}_{\Sigma\theta} & \mathcal{J}_\Sigma \end{pmatrix} \begin{pmatrix} \mathcal{I}_\theta^{-1} & 0 \\ 0 & \mathcal{I}_\Sigma^{-1} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_\theta^{-1} \mathcal{J}_\theta \mathcal{I}_\theta^{-1} & \mathcal{I}_\theta^{-1} \mathcal{J}_{\theta\Sigma} \mathcal{I}_\Sigma^{-1} \\ \mathcal{I}_\Sigma^{-1} \mathcal{J}_{\Sigma\theta} \mathcal{I}_\theta^{-1} & \mathcal{I}_\Sigma^{-1} \mathcal{J}_\Sigma \mathcal{I}_\Sigma^{-1} \end{pmatrix}.$$

We can see the asymptotic covariance matrix for $\hat{\theta}$ is just $\mathcal{I}_\theta^{-1} \mathcal{J}_\theta \mathcal{I}_\theta^{-1}$. \square

